

ODD H-DEPTH AND H-SEPARABLE EXTENSIONS

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ABSTRACT. Let $C_n(A, B)$ be the relative Hochschild bar resolution groups of a subring $B \subseteq A$. The subring pair has right depth $2n$ if $C_{n+1}(A, B)$ is isomorphic to a direct summand of a multiple of $C_n(A, B)$ as A - B -bimodules; depth $2n + 1$ if the same condition holds only as B - B -bimodules. It is then natural to ask what is defined if this same condition should hold as A - A -bimodules, the so-called H-depth $2n - 1$ condition. In particular, the H-depth 1 condition coincides with A being an H-separable extension of B . In this paper the H-depth of semisimple subalgebra pairs is derived from the transpose inclusion matrix, and for QF extensions it is derived from the odd depth of the endomorphism ring extension. For general extensions characterizations of H-depth are possible using the H-equivalence generalization of Morita theory.

1. INTRODUCTION

Given a unital subring B in an associative unital ring A where $1_A = 1_B$, this paper continues a study of certain bimodule conditions on the n -fold tensor products $A \otimes_B \cdots \otimes_B A$. In the papers [2, 1, 17] the ring extension $A \supseteq B$ is said to have left depth 2, right depth 2, or depth 3 if the tensor-square has a split bimodule monomorphism into a multiple of A as respectively B - A -, A - B - or B - B -bimodules. The depth 2 conditions are interesting from the point of view of Galois theory, since $\text{End}_B A_B$ has a finite projective bialgebroid structure over the centralizer subring A^B and acts naturally on A (e.g., [17, 22]). The depth 3 condition on a Frobenius extension $A|B$ is also of Galois-theoretic interest, since the left regular representation $\lambda : A \hookrightarrow \text{End}_B A_B := E$ restricts to a ring extension $B \hookrightarrow E$ having depth 2 [21, Theorem 2.5]. In this case the ring $\text{End}_B E_A$ is a left coideal subring of $\text{End}_B E_B$ with good Galois-theoretic properties of a “depth-3 tower” $B \subseteq A \hookrightarrow E$ sketched in [21, Sections 4,5].

A similar definition of the ring extension $A \supseteq B$ having left depth $2n$, right depth $2n$ or depth $2n + 1$ holds: there is a split monic of the $(n + 1)$ -fold tensor product into a multiple of the n -fold tensor product as natural B - A -, A - B - or B - B -bimodules, respectively [1]. In case this is a Frobenius extension with surjective Frobenius homomorphism $E : A \rightarrow B$, the ring extension having depth n embeds in a tower of iterated right endomorphism rings $E_1 \hookrightarrow E_2 \hookrightarrow \cdots$ where $B \hookrightarrow E_{n-3} \hookrightarrow E_{n-2}$ is also a “depth-3 tower” [23].

The minimum depth $d(B, A)$ realizes each positive integer for complex semisimple algebras $B \subseteq A$ with Bratteli diagram a Dynkin diagram of type A_n ; see [8, 3.11]. However, for the group algebras $B = \mathbb{C}[H]$ and $A = \mathbb{C}[G]$ of a subgroup H of a finite group G , the values of $d(B, A)$ seem to be limited to the odd values and only the even values $\{2, 4, 6\}$; see [1, 2, 8, 9, 10].

In the three bimodule-theoretic definitions above of left, right even depth and odd depth, the fourth case of A - A -bimodules has been sidestepped so far, but is taken

up in this paper. We pose the question, what is defined on a ring extension $A \supseteq B$ if the $(n+1)$ -fold tensor product has a split A - A -bimodule monic into a multiple of the n -fold tensor product $A \otimes_B \cdots \otimes_B A$? This question has classical roots in case $n = 1$; the condition that $A \otimes_B A$ is a direct summand of $A^n = A \oplus \cdots \oplus A$ (or nA in additive notation) as natural A -bimodules is the condition that A is an H-separable extension of B [12]. These have an elaborate theory generalizing Azumaya algebra, where among other things one proves with some commutative algebra that also ${}_A A_A$ is a direct summand of $A \otimes_B A$, i.e. A is a separable extension of B . H-separability is studied in e.g. [12, 11, 15, 16, 19].

In section 2 we define $A \supseteq B$ having H-depth $2n-1$ as the condition just given in terms of A -bimodules on the $(n+1)$ -fold and n -fold tensor products. We sketch the general theory for ring extensions, noting that the minimum H-depth $d_H(B, A)$ and minimum depth $d(B, A)$ differ by at most 2, if one is finite. We also apply results from [1, 23] about when $d(B, A)$ is finite. In section 3 we restrict to A and B being complex semisimple algebras when information about the subalgebra structure is nicely recorded by weighted bicolored multi-graphs and inclusion matrices that are viewable as homomorphisms of the K -groups $K_0(B) \rightarrow K_0(A)$. In this case the H-depth is a condition on the transpose of the inclusion matrix. In section 4 we note that a Frobenius extension $A \supseteq B$ having H-depth $2n-1$ occurs precisely when the left regular extension $E \supseteq A$ (seen above) has depth $2n-1$. However we note through examples that the depths $d_H(B, A)$, $d(B, A)$ and $d(A, E)$ may differ from one another.

2. GENERAL THEORY

Given a unital associative ring R and unital R -modules M and N , we write $N \oplus * \cong M^q$ if N is isomorphic to a direct summand in $M^q = M \oplus \cdots \oplus M$ (q times). Recall that if there is also a positive integer s such that $M \oplus * \cong N^s$, then M and N are similar, or H-equivalent, as R -modules; denoted by $M \sim N$, indeed an equivalence relation. In this case their endomorphism rings $\text{End } M_R$ and $\text{End } N_R$ are Morita equivalent with Morita context bimodules $\text{Hom}(M_R, N_R)$ and $\text{Hom}(N_R, M_R)$ (with composition as module actions and Morita pairings). If the category of finitely generated R -modules has unique factorization into indecomposables, then finitely generated M and N have the same indecomposable constituents if and only if M and N are H-equivalent modules. If F is an additive endofunctor of the category of R -modules, then $M \sim N$ implies $F(M) \sim F(N)$; which in practice means that H-equivalent bimodules may replace one another in certain H-equivalences of tensor products.

Throughout this paper, let A be a unital associative ring and $B \subseteq A$ a subring where $1_B = 1_A$. Note the natural bimodules ${}_B A_B$ obtained by restriction of the natural A - A -bimodule (briefly A -bimodule) A , also to the natural bimodules ${}_B A_A$, ${}_A A_B$ or ${}_B A_B$, which are referred to with no further notation. Equivalently we denote the proper ring extension $A \supseteq B$ occasionally by $A|B$. (Often results are valid as well for a ring homomorphism $B \rightarrow A$ and its induced bimodules on A .)

Let $C_0(A, B) = B$, and for $n \geq 1$,

$$C_n(A, B) = A \otimes_B \cdots \otimes_B A \quad (n \text{ times } A)$$

For $n \geq 1$, the $C_n(A, B)$ has a natural A -bimodule structure given by $a(a_1 \otimes \cdots \otimes a_n)a' = aa_1 \otimes \cdots \otimes a_na'$. Of course, this bimodule structure restricts to B - A -, A - B - and B -bimodule structures as we may need them.

Definition 2.1. *The subring $B \subseteq A$ has H-depth $2n - 1 \geq 1$ if for some positive integer q ,*

$$(1) \quad C_{n+1}(A, B) \oplus * \cong C_n(A, B)^q.$$

Note that if a subring has H-depth $2n - 1$, it has H-depth $2n + 1$ by tensoring the displayed equation by either $-\otimes_B A_A$ or ${}_A A \otimes_B -$. If $B \subseteq A$ has a finite H-depth, denote its minimum H-depth by $d_H(B, A)$.

Note that H-depth 1 occurs when ${}_A A \otimes_B A_A \oplus * \cong {}_A A_A^n$ for some $n \in \mathbb{Z}_+$, which is the condition that A be an H-separable extension of B . (Replacing A^n with the direct sum $A^{(I)}$ for I an arbitrary indexing set in the definition of H-separable extension does not obtain anything different since $A \otimes_B A$ is a cyclic A -bimodule [22, Prop. 1.3(4)].) This is a classical generalization of the Azumaya condition on algebras to ring extensions (see for example [12, 11, 15, 16, 19]); indeed, one may prove that A is a separable extension of B , so that ${}_A A_A \oplus * \cong {}_A A \otimes_B A_A$ since the multiplication mapping $\mu : A \otimes_B A \rightarrow A$ splits. Recall too that a ring extension $A|B$ is separable if and only if the relative Hochschild cohomology groups $H^n(A, B; M) = 0$ for all $n > 0$ and all bimodule coefficient modules ${}_A M_A$ [11].

Lemma 2.2. *A subring $B \subseteq A$ has H-depth $2n - 1$ if and only if $C_n(A, B) \sim C_{n+1}(A, B)$ as A -bimodules ($n = 1, 2, 3, \dots$).*

Proof. We just noted above that the H-depth 1 condition is H-separability, which implies separability, and the two conditions together give $C_2(A, B) \sim C_1(A, B)$ as A -bimodules. For $n = 2$ the A -bimodule $\text{epi } C_3(A, B) \rightarrow C_2(A, B)$ given by $a_1 \otimes a_2 \otimes a_3 \mapsto a_1 a_2 \otimes a_3$ is split by $a_1 \otimes a_2 \mapsto a_1 \otimes 1 \otimes a_2$. Then $C_2(A, B) \oplus * \cong C_3(A, B)$. An obvious generalization gives that $C_n(A, B) \oplus * \cong C_{n+1}(A, B)$. It follows that the condition in the lemma is equivalent to the condition in the definition. \square

Let $C_0(A, B)$ denote the natural B -bimodule B itself. Recall from [1, 23] that a subring $B \subseteq A$ has right depth $2n$ if

$$(2) \quad C_{n+1}(A, B) \sim C_n(A, B)$$

as natural A - B -bimodules; left depth $2n$ if the same condition holds as B - A -bimodules; if both left and right conditions hold, it has depth $2n$; and depth $2n + 1$ if the same condition holds as B -bimodules. Note that if the subring has left or right depth $2n$, it automatically has depth $2n + 1$ by restriction to B -bimodules. Also note that if the subring has depth $2n + 1$, it has depth $2n + 2$ by tensoring the H-equivalence by $-\otimes_B A$ or $A \otimes_B -$. The *minimum depth* is denoted by $d(B, A)$; if $B \subseteq A$ has no finite depth, write $d(B, A) = \infty$.

The dependence of depth and H-depth only on the H-equivalence class of the natural bimodule of a ring extension is made explicit below.

Lemma 2.3. *Suppose $A \supseteq C$ and $B \supseteq C$ are two ring extensions of the same ring. If the natural C -bimodules are H-equivalent, ${}_C A_C \sim {}_C B_C$, then $A \supseteq C$ has depth $2n + 1$ if and only if $B \supseteq C$ has depth $2n + 1$. Suppose moreover $A \supseteq B \supseteq C$ is a tower of rings. If $B \supseteq C$ has H-depth $2n - 1$ (for $n > 1$), then $A \supseteq C$ has H-depth $2n - 1$.*

Proof. By the substitution principle for the H-equivalent bimodules ${}_C A_C \sim {}_C B_C$, we obtain $C_n(A, C) \sim C_n(B, C)$ as C -bimodules for all $n \geq 1$. Thus, $C_{n+1}(A, C) \sim C_n(A, C)$ if and only if $C_{n+1}(B, C) \sim C_n(B, C)$ for all $n \geq 0$. This proves the first statement in the lemma.

The second statement follows from applying the additive functor $A \otimes_B - \otimes_B A$ (from B -bimodules into A -bimodules) to the H-equivalence of B -bimodules, $C_{n+1}(B, C) \sim C_n(B, C)$, cancelling certain trivial tensors to obtain $A \otimes_C C_{n-1}(B, C) \otimes_C A \sim A \otimes_C C_{n-2}(B, C) \otimes_C A$ as A -bimodules. But $C_n(B, C) \sim C_n(A, C)$ as C -bimodules for each $n > 0$ by the hypothesis, so that $C_{n+1}(A, C) \sim C_n(A, C)$ as A -bimodules. Thus $A \supseteq C$ has H-depth $2n - 1$ as well. \square

Proposition 2.4. *If a subring has H-depth $2n - 1$, then it has depth $2n$. If a subring has left or right depth $2n$, then it has H-depth $2n + 1$. As a consequence,*

$$(3) \quad |d(B, A) - d_H(B, A)| \leq 2.$$

Proof. The first statement follows from restricting the condition on A -bimodules for H-depth in the lemma to the condition (2) on either A - B - or B - A -bimodules. The second statement follows for example by tensoring the right depth $2n$ condition from the right by $-\otimes_B A_A$ to obtain the H-depth condition. The inequality follows from applying the first two statements. \square

For example, an H-depth 1 extension is known to have depth 2, with associated Galois structure computed in [19].

Example 2.5. Let $B = \mathbb{C}^2 \hookrightarrow A = M_2(\mathbb{C})$ be given by $(\lambda, \mu) \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ for each $\lambda, \mu \in \mathbb{C}$. Since B is a semisimple subalgebra of the Azumaya algebra A , this is an H-separable extension. The extension is also normal and depth 2 by [8, Prop. 4.3]. By the results of [6] a subalgebra pair of complex semisimple algebras $B \subseteq A$ has depth 1 if and only if their centers satisfy $\mathcal{Z}(B) \subseteq \mathcal{Z}(A)$. In this example this is not the case, whence $d_H(B, A) = 1$ and $d(B, A) = 2$.

In Section (3) it is noted how to compute depth and H-depth directly from the inclusion matrix $M = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and its transpose.

For a k -algebra B let B^e denote $B \otimes_k B^{\text{op}}$.

Corollary 2.6. *Let $B \subseteq A$ be a subring pair of finite dimensional algebras. If either B^e has finite representation type or both A and B are group algebras $k[G]$ and $k[J]$ with $J < G$ a subgroup pair, then $d_H(B, A) < \infty$.*

Proof. If B^e has finite representation type, it is shown in [23] that subring depth $d(B, A)$ is bounded by one plus twice the number of isomorphism classes of indecomposable B -bimodules. If $A \supseteq B$ is a group algebra extension, it is shown in [1] that $d(B, A)$ is bounded by twice the index of the normalizer of J in G . \square

Recall the relative Hochschild cochain groups with coefficient bimodule ${}_A M_A$ [13]: they are denoted and defined by $C^n(A, B; M) = \text{Hom}_{B^e}(C_n(A, B), M)$.

Proposition 2.7. *Suppose $A \supseteq B$ has H-depth $2n - 1 \geq 3$. Then $C^{n-1}(A, B; M) \sim C^{n-2}(A, B; M)$ as abelian groups.*

Proof. Apply the additive functor $\text{Hom}_{A^e}(-, M)$ to the H-equivalence in the lemma above. Note that $\text{Hom}_{A^e}(C_{n+1}(A, B), M) \cong \text{Hom}_{B^e}(C_{n-1}(A, B), M)$. The H-equivalence in the proposition follows. \square

This may be applied to H-depth 3 and the natural bimodule $M = A$ to obtain the following.

Corollary 2.8. *For an H-depth 3 extension $A \supseteq B$, the endomorphism ring $\text{End}_B A_B$ is H-equivalent to the centralizer A^B as modules over the center of A .*

We end with a characterization of ring extensions $A|B$ having H-depth 3. Let $T := (A \otimes_B A)^B$ and note $T \cong \text{End}_A A \otimes_B A_A$ via $t \mapsto (a \otimes a' \mapsto ata')$ with inverse $F \mapsto F(1 \otimes 1)$. Thus T is a ring with multiplication $st = t^1 s^1 \otimes s^2 t^2$ and $1_T = 1 \otimes 1$, and $A \otimes_B A = C_2(A, B)$ is a left T -module ($t \cdot (a \otimes a') = ata'$); in fact a T - A^e -bimodule.

In addition $Q := (A \otimes_B A \otimes_B A)^B$ is a right T -module via $q \cdot t = t^1 q t^2$. It is a generator T -module via an obvious split epi $Q_T \rightarrow T_T$.

Theorem 2.9. *A ring extension $A|B$ has H-depth 3 if and only if Q_T is finite generated projective and the mapping $q \otimes_T (a \otimes a') \mapsto aqa'$ is an isomorphism of A -bimodules,*

$$Q \otimes_T C_2(A, B) \xrightarrow{\cong} C_3(A, B)$$

Proof. (\Leftarrow) From $Q_T \oplus * \cong T_T^n$ and tensoring from the right by $- \otimes_T C_2(A, B)$, one obtains the H-depth 3 defining condition

$$(4) \quad C_3(A, B) \oplus * \cong C_2(A, B)^n$$

as A -bimodules.

(\Rightarrow) Applying a diagram chase to (1) where $n = 2$, similar to the one for extracting dual bases for a projective module, one finds elements $q_i \in Q$ and $g_i \in \text{Hom}({}_B A_B, {}_B A \otimes_B A_B)$ ($i = 1, \dots, q$) such that for all $a_1, a_2, a_3 \in A$,

$$a_1 \otimes_B a_2 \otimes_B a_3 = \sum_{i=1}^q a_1 g_i^1(a_2) q_i g_i^2(a_2) a_3$$

where $g_i(a) = g_i^1(a) \otimes_B g_i^2(a)$ is a type of Sweedler notation suppressing a possible summation.

It follows that the mapping defined in the theorem has inverse mapping defined on $c \in C_3(A, B)$ by

$$c \mapsto \sum_{i=1}^q q_i \otimes_T c^1 g_i(c^2) c^3$$

where we again use Sweedler-type notation for c .

(Alternatively the mapping in the theorem is an isomorphism by applying [15, Theorem 2.20]; it is in particular the left vertical isomorphism in Figure 1.)

Finally $Q \cong \text{Hom}({}_A C_2(A, B)_A, {}_A C_3(A, B)_A)$ via $q \mapsto (a \otimes a' \mapsto aqa')$ with inverse $G \mapsto G(1 \otimes 1)$. But this Hom-group is a Morita context bimodule for the Morita equivalent rings $\text{End}_A C_2(A, B)_A \cong T$ and $\text{End}_A C_3(A, B)_A$ stemming from the H-equivalence of $C_2(A, B)$ and $C_3(A, B)$. Whence Q_T is finite projective. \square

Similar characterizations of (H-) separable extensions are obtained in [18, Theorems 4.1, 4.2].

Given a ring extension $A|B$, let E denote its right endomorphism ring $\text{End } A_B$. Its natural A -bimodule structure is given by $(afa')(x) = af(a'x)$. Recall that a right B -module V coinduces to an A -module $\text{Hom}(A_B, V_B)$; e.g., $V = \text{Res}_B^A A$ has coinduced module E_A .

Proposition 2.10. *Suppose $A|B$ has finitely generated projective module A_B . Then $d_H(B, A) \leq 3$ if and only if $\text{CoInd}_B^A \text{Res}_B^A E \oplus * \cong E^n$ as A -bimodules for some $n \in \mathbb{Z}_+$.*

Proof. (\Rightarrow) This direction in the proof does not require finite projectivity of A_B . Apply the additive endofunctor $\text{Hom}(-_A, A_A)$ to the isomorphism (4) in the category of A -bimodules. Note that $\text{Hom}(A \otimes_B A_A, A_A) \cong E$ via $F \mapsto F(- \otimes_B 1_A)$, and that $\text{Hom}(C_3(A, B)_A, A_A) \cong \text{Hom}(A \otimes_B A_B, A_B)$ via $F \mapsto F(- \otimes_B - \otimes_B 1_A)$. This obtains

$$(5) \quad \text{Hom}(A \otimes_B A_B, A_B) \oplus * \cong E^n$$

as natural A -bimodules.

By the adjoint relation between homming and tensoring, $\text{Hom}(A \otimes_B A_B, A_B) \cong \text{Hom}(A_B, E_B)$ via $F \mapsto (a \mapsto F(a \otimes -))$. The A -bimodule isomorphism in the theorem follows.

(\Leftarrow) If A_B is finite projective, then $A \otimes_B A_A$ is also finite projective, therefore reflexive. Then applying $\text{Hom}(_A -, _A A)$ to the isomorphism (5) obtains the defining eq. (4) for H-depth 3 extension $A|B$. \square

3. SUBALGEBRA PAIRS OF COMPLEX SEMISIMPLE ALGEBRAS

Given a matrix M we let M^t denote its transpose matrix in this section. Two r -by- s matrices M, N satisfy an inequality $M \leq N$ if each pair of (i, j) -entries satisfy $M_{ij} \leq N_{ij}$. The $r \times s$ zero matrix is denoted by 0.

Let $B \subseteq A$ be a subring pair of semisimple complex algebras. Then the minimum depth $d(B, A)$ may be computed from the inclusion matrix M , equivalently an r -by- s induction-restriction table of r B -simples induced to non-negative integer linear combination of s A -simples along rows; by Frobenius reciprocity, columns show restriction of A -simples in terms of B -simples. The procedure to obtain $d(B, A)$ given in the paper [8] is the following: let $M^{[2n]} = (MM^t)^n$ and $M^{[2n+1]} = M^{[2n]}M$ (and $M^{[0]} = I_r$), then the matrix M has depth $n \geq 1$ if for some $q \in \mathbb{Z}_+$

$$(6) \quad M^{[n+1]} \leq qM^{[n-1]}$$

Note that if M has depth n , it has depth $n+1$ by multiplying the inequality by $M \geq 0$. The minimum depth of M is equal to $d(B, A)$ (see [1]). One may note that $d(B, A) \leq 2d - 1$ where MM^t has minimal polynomial of degree d [8]. Thus $d(B, A) < \infty$ from which it follows from inequality (3) that $d_H(B, A) < \infty$ for a complex semisimple subalgebra pair $B \subseteq A$; alternatively, note that B^e has finite representation type and apply Corollary (2.6)

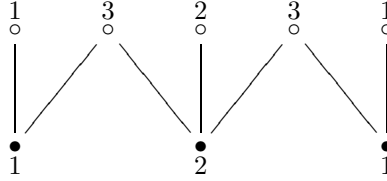
Let $Z(M)$ be the number of zero entries in an $r \times s$ matrix M of nonnegative integers: since an inclusion matrix M has nonzero rows and columns, $rs - \max\{r, s\} \geq Z(M) \geq 0$. Since $M \geq 0$, it follows that $Z(M^{[n]})$ are decreasing nonnegative integers with increasing odd or even bracketed powers of M :

$$Z(M^{[n+2]}) \leq Z(M^{[n]}).$$

It is quite easy to see that $d(B, A)$ is the least n for which $Z(M^{[n+2]}) = Z(M^{[n]})$ where M is the inclusion matrix for $B \subseteq A$ (cf. [8]).

In terms of the bipartite graph of the inclusion $B \subseteq A$, $d(B, A)$ is the lesser of the minimum odd depth and the minimum even depth [8]. The matrix M is an incidence matrix of this bipartite graph if all entries greater than 1 are changed to 1, while zero entries are retained as 0: let the B -simples be represented by r black dots in a bottom row of the graph, and A -simples by s white dots in a top row, connected by edges joining black and white dots (or not) according to the 0-1-matrix entries obtained from M . The minimum odd depth of the bipartite graph is 1 plus the diameter in edges of the row of black dots (indeed an odd number) [8, 3.6], while the minimum even depth is 2 plus the largest of the diameters of the bottom row where a subset of black dots under one white dot is identified with one another [8, 3.10].

Example 3.1. Let $A = \mathbb{C}[S_4]$, the complex group algebra of the permutation group on four letters, and $B = \mathbb{C}[S_3]$. The inclusion diagram pictured below with the degrees of the irreducible representations, is determined from the character tables of S_3 and S_4 or the branching rule (for the Young diagrams labelled by the partitions of n and representing the irreducibles of S_n).



This graph has minimum odd depth 5 and minimum even depth 6, whence $d(B, A) = 5$. Alternatively, the inclusion matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

satisfies the inequality (6) only when $n \geq 5$.

Recall orthonormal expansion of a vector in an inner product space with orthonormal bases e_1, \dots, e_n : $v = \sum_{i=1}^n \langle v, e_i \rangle e_i$.

Theorem 3.2. Suppose $B \subseteq A$ is a subalgebra pair of complex semisimple algebras with inclusion matrix M . Then $B \subseteq A$ has H-depth $2n - 1$ iff the symmetric matrix $S = M^t M$ satisfies

$$(7) \quad S^n \leq qS^{n-1}$$

for some $q \in \mathbb{Z}_+$.

Proof. (\Rightarrow) Suppose the isomorphism classes of simple right A -modules are represented by $\text{Irr}(A) = \{V_1, \dots, V_s\}$, and $\text{Irr}(B) = \{W_1, \dots, W_r\}$. Then M is an $r \times s$ -matrix with entries m_{ij} such that restriction of modules, $V_j \downarrow_B \cong \oplus_{i=1}^r m_{ij} W_i$, and induction of modules, $W_i \uparrow^A \cong \oplus_{j=1}^s m_{ij} V_j$ in additive notation. With the usual inner product of A - or B -modules, where $\langle W_i, W_j \rangle = \delta_{ij} = \langle V_i, V_j \rangle$, we note that

$m_{ij} = \langle V_j \downarrow_B, W_i \rangle_B$ and

$$(8) \quad (M^t M)_{ij} = \sum_k m_{ki} m_{kj} = \sum_{k,t} m_{ki} m_{tj} \langle W_k, W_t \rangle \\ = \langle V_i \downarrow_B, V_j \downarrow_B \rangle_B = \langle V_i \downarrow_B \uparrow^A, V_j \rangle_A$$

the last step being Frobenius reciprocity.

Now if ${}_A A \otimes_B {}_A A \oplus * \cong {}_A A_A^q$, then $V_i \otimes_B {}_A A \oplus * \cong qV_i$ after standard tensor cancellations. Thus, $\text{Ind}_B^A \text{Res}_B^A V_i$ is isomorphic to a direct summand of a multiple of V_i . Denoting the endo-operator on A -modules by $T = \text{Ind}_B^A \text{Res}_B^A$, we similarly see that the H-depth $2n - 1$ condition in Definition (1) implies that $T^n(V_i) \oplus * \cong qT^{n-1}(V_i)$. Thus for each V_j there is the inequality

$$(9) \quad \langle T^n(V_i), V_j \rangle \leq q \langle T^{n-1}(V_i), V_j \rangle.$$

But the matrix of T in terms of the basis $\{V_1, \dots, V_s\}$ is given by $S = M^t M$ according to the computation in Eq. (8). Thus, the inequality (9) becomes $(S^n)_{ij} \leq q(S^{n-1})_{ij}$ for all $i, j = 1, \dots, s$.

(\Leftarrow) If the inclusion matrix M of semisimple subalgebra pair $B \subseteq A$ satisfies $(M^t M)^n \leq q(M^t M)^{n-1}$ for some $q \in \mathbb{Z}_+$, then $\langle T^n(V_i), V_j \rangle \leq q \langle T^{n-1}(V_i), V_j \rangle$ for all A -simples V_j . Via unique module decomposition into simples, we find a monic natural transformation $T^n \hookrightarrow qT^{n-1}$ of endofunctors on the category $A\text{-Mod}$. Now A and therefore A^e are separable \mathbb{C} -algebras, so as in [[18], Theorem 2.1(6), pp. 3107-3108], we apply the natural monic to the right regular module A_A , apply the natural transformation property to all left multiplications λ_a ($a \in A$), and note that $C_{n+1}(A, B) \hookrightarrow qC_n(A, B)$ splits by Maschke as an A -bimodule monic. Hence A has H-depth $2n - 1$ over its subalgebra B . \square

Following [8] we say that an $r \times s$ -matrix M of nonnegative integers, and nonzero rows and columns, has depth n if the inequality (6) is satisfied. Similarly define the minimum depth $d(M)$ to be the least positive integer n for which M has depth n . The matrix M always has a finite depth, bounded by degree of the minimum polynomial of the symmetric matrix MM^t .

Corollary 3.3. *With the hypotheses of the theorem, the minimum H-depth of $B \subseteq A$ and the minimum depth of the transpose inclusion matrix satisfy*

$$(10) \quad 0 \leq d_H(B, A) - d(M^t) \leq 1.$$

Proof. The matrix M^t is the inclusion matrix for the endomorphism ring extension $A \hookrightarrow E$ (given by $a \mapsto \lambda_a$ where $\lambda_a(x) = ax$ for all $a, x \in A$), also a subalgebra pair of complex semisimple algebra inclusions [8, 3.13 and above]. Note that the inequality (7) is the depth $2n - 1$ condition on the transpose matrix M^t . Thus if $d(M^t)$ is odd, then $d_H(B, A) = d(M^t)$, and if $d(M^t)$ is even, then $d_H(B, A) = d(M^t) + 1$. \square

It follows from [8, Prop. 2.5] that for any inclusion matrix M ,

$$(11) \quad |d(M^t) - d(M)| \leq 1$$

Example 3.4. Continuing the example $B = \mathbb{C}[S_3] \subseteq \mathbb{C}[S_4] = A$ above, where we obtained M and its depth $d(M) = d(B, A) = 5$, we may continue computing bracketed powers of M^t to obtain $d(M^t) = 6$ and $d_H(B, A) = 7$. (Alternatively,

the depth $d(M^t) = d(A, E)$ may be computed as the depth of the graph in Example (3.1) reflected about the top row of white dots (so the 3 black dots land above the 5 white dots with weights 4, 8 and 4. See [8, 3.12] for the explanation in terms of Morita equivalence.) This shows that the various inequalities in (3), (10), (11) and (13) may not be improved.

Example 3.5. Let H_8 denote the eight-dimensional self-dual semisimple Hopf algebra of Masuoka [24] and Kac-Paljutkin [14] over an algebraically closed field k of characteristic zero with generators x, y, z and relations $x^2 = y^2 = 1$, $xy = yx$, $zx = yz$, $zy = xz$ and $2z^2 = 1 + x + y - xy$. Its coalgebra structure is determined by $\Delta(x) = x \otimes x$, $\varepsilon(x) = 1$, $S(x) = x$, $\Delta(y) = y \otimes y$, $\varepsilon(y) = 1$, $S(y) = y$, and $\Delta(z) = \frac{1}{2}((1 + y) \otimes 1 + (1 - y) \otimes x)(z \otimes z)$, $\varepsilon(z) = 1$ and $S(z) = z$.

Burciu [5] computes the irreducible characters of H_8 and its Drinfeld double $D(H_8)$ as well as the induced representations. The space of irreducible characters of H_8 is 5-dimensional with four linear and one degree 2 irreducible characters: $H_8 \cong k^4 \times M_2(k)$. The space of irreducible characters of $D(H_8)$ is 22-dimensional with $D(H_8) \cong k^8 \times M_2(k)^{14}$ [5, pp. 501-503]. Thus, M is a 5×14 matrix determined by the table [5, p. 503]; its bracketed square is computed to be

$$MM^t = \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ 1 & 5 & 1 & 1 & 0 \\ 1 & 1 & 5 & 1 & 0 \\ 1 & 1 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$$

From this matrix and the table it follows that $d(H_8, D(H_8)) = 3$. The matrix $M^t M$ is an order 22 square matrix \mathcal{S} that does not satisfy $Z(\mathcal{S}^2) = Z(\mathcal{S})$, whence $d_H(H_8, D(H_8)) = 5$.

4. FROBENIUS EXTENSIONS

A Frobenius extension $A \supseteq B$ is characterized by any of the following four conditions [15]. First, that A_B is finite projective and ${}_B A_A \cong \text{Hom}(A_B, B_B)$. Secondly, that ${}_B A$ is finite projective and ${}_A A_B \cong \text{Hom}({}_B A, {}_B B)$. Thirdly, that coinduction and induction of right (or left) B -modules is naturally equivalent. Fourth, there is a Frobenius coordinate system $(E : A \rightarrow B; x_1, \dots, x_m, y_1, \dots, y_m \in A)$, which satisfies

$$E \in \text{Hom}({}_B A_B, {}_B B_B), \quad \sum_{i=1}^m E(ax_i)y_i = a = \sum_{i=1}^m x_i E(y_i a) \quad (\forall a \in A).$$

The so-called dual bases equations may be used to show that $\sum_i x_i \otimes y_i \in (A \otimes_B A)^A$ [15]. A Frobenius extension $A \supseteq B$ has generator module A_B iff it has a surjective Frobenius homomorphism $E : A \rightarrow B$ [23]. Well-known examples of Frobenius extensions are group algebra extensions, unimodular Hopf algebra extensions, and projective subalgebra pair of symmetric algebras [15].

More generally a ring extension $A \supseteq B$ is a *QF extension* if both A_B and ${}_B A$ are finite projective, and the natural bimodules are H-equivalent: ${}_A A_B \sim {}_A \text{Hom}({}_B A, {}_B B)_B$ and ${}_B A_A \sim {}_B \text{Hom}(A_B, B_B)_A$ [26]. A Frobenius extension $A \supseteq B$ is a QF extension since it is left and right finite projective and satisfies the stronger conditions that A is *isomorphic* to its right B -dual A^* and its left B -dual *A as natural B - A -bimodules, respectively A - B -bimodules; the more precise

definition are given in the next section. QF extensions that are not Frobenius extensions may be found among weak Hopf algebras over their separable base algebras, and matrix examples by Morita [25].

A Frobenius (or QF) extension $A \supseteq B$ enjoys an *endomorphism ring theorem* [26, 25], which shows that $E := \text{End } A_B \supseteq A$ is a Frobenius (respectively, QF) extension, where the default ring homomorphism $A \rightarrow E$ is understood to be the left multiplication mapping $\lambda : a \mapsto \lambda_a$ where $\lambda_a(x) = ax$. It is worth noting that λ is a left split A -monomorphism (by evaluation at 1_A) so ${}_A E$ is a generator.

The endomorphism ring E is isomorphic to $A \otimes_B A$ as natural A -bimodules via $f \mapsto \sum_{i=1}^m f(x_i) \otimes_B y_i$ with inverse $a \otimes_B a' \mapsto \lambda_a \circ E \circ \lambda_{a'}$ where λ_x denotes left multiplication on A by an element $x \in A$. More generally, coinduction is naturally isomorphic to induction of B -modules to A -modules; so that given a right B -module V , there are similarly defined isomorphisms $V \otimes_B A \cong \text{Hom}(A_B, V_B)$. For a QF extension $A|B$, the isomorphism is replaced with an H-equivalence: $V \otimes_B A \sim \text{Hom}(A_B, V_B)$ [23]. An immediate consequence of this is a specialization of Proposition (2.10) to QF extensions with a different proof.

Proposition 4.1. *Suppose $A|B$ is a QF extension, Then $A|B$ has H-depth 3 if and only if $E \otimes_B A \sim E$ as A -bimodules (alternatively, $\text{Ind}_B^A \text{Res}_B^A E \sim E$ as A -bimodules).*

Proof. This follows from $E \sim A \otimes_B A$ as A -bimodules and Lemma (2.2) in the case $n = 2$. \square

Theorem 4.2. *Suppose $A|B$ is a Frobenius extension or a QF extension. Then $A|B$ has H-depth $2n - 1$ if and only if the endomorphism ring extension $E|A$ has depth $2n - 1$.*

Proof. If $A|B$ is a Frobenius extension, we have $E \cong A \otimes_B A$ as natural A -bimodules. Then

$$(12) \quad C_n(E, A) \cong (A \otimes_B A) \otimes_A \cdots \otimes_A (A \otimes_B A)$$

(n times E and $n - 1$ times A), so that $C_n(E, A) \cong C_{n+1}(A, B)$ since there remain $2n - (n - 1)$ A 's after a standard tensor cancellation. Thus as A -bimodules $C_{n+1}(A, B) \sim C_n(A, B)$ if and only if $C_n(E, A) \sim C_{n-1}(E, A)$ as A -modules, the latter being condition (2) for $E|A$ having depth $2n - 1$.

If $A|B$ is QF extension, then $A \otimes_B A \sim E$ as A -bimodules [23], and the same proof carries through with isomorphisms replaced by H-equivalences. \square

Since H-depth only assumes odd integer values, it follows from the theorem that

$$(13) \quad |d_H(B, A) - d(A, E)| \leq 1.$$

Recall that a ring extension $A|B$ is centrally projective if ${}_B A_B \oplus * \cong {}_B B_B^n$ for some $n \in \mathbb{Z}_+$, and a split extension if ${}_B B_B \oplus * \cong {}_B A_B$. The ring extension $A|B$ has depth 1 iff it is a centrally projective, split extension (cf. [23, 3]).

Corollary 4.3. *A Frobenius extension is H-separable if and only if its right endomorphism ring extension has depth 1.*

It was noted in [20] that a group algebra extension $A = \mathbb{C}G \supseteq \mathbb{C}J = B$ that is H-separable is necessarily trivial: $G = J$. In [18] it was proven that group algebra extension $\mathbb{C}G \supseteq \mathbb{C}J$ has depth 2 if and only if J is a normal subgroup of G (and the same result holds for any base ring by [1]). Again let $E = \text{End } A_B$.

Corollary 4.4. *Suppose J is a proper normal subgroup of a finite group G . Then $d(A, E) = 2$*

With the Sweedler A -coring $A \otimes_B A$ of a ring extension $A | B$ in mind, eq. (12) suggests a definition of H-depth of an A -coring \mathcal{C} that generalizes H-depth of a ring extension. An A -coring \mathcal{C} has H-depth $2n - 1$ if $\mathcal{C}^{\otimes_A n} \sim \mathcal{C}^{\otimes_A n-1}$ as A -bimodules for $n \geq 1$ and $\mathcal{C}^0 = A$.

Depth of an A -coring \mathcal{C} with grouplike $g \in \mathcal{C}$ is similarly defined. Let $B = \mathcal{C}^g$ be the invariant subring of A . Then \mathcal{C} has depth $2n + 1$, left depth $2n$ or right depth $2n$ if $\mathcal{C}^{\otimes_A n} \sim \mathcal{C}^{\otimes_A n-1}$ as respectively B - B -, B - A - or A - B -bimodules, where $\mathcal{C}^{-1} = B$. The theory of corings, grouplikes, Sweedler corings and ring extensions having depth 2 are to be found in the book [4]. Depth and H-depth of corings will be investigated in a future paper.

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